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Painlevé transcendent describes quantum correlation function of the *XXZ* antiferromagnet away from the free-fermion point

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Abstract. We consider quantum correlation functions of the antiferromagnetic spin-$\frac{1}{2}$ Heisenberg *XXZ* spin chain in a magnetic field. We show that for a magnetic field close to the critical field h_c (for the critical magnetic field the ground state is ferromagnetic) certain correlation functions can be expressed in terms of the solution of the Painlevé V transcendent. This establishes a relation between solutions of Painlevé differential equations and quantum correlation functions in models of *interacting* fermions. Painlevé transcendents were known to describe correlation functions in models with free fermionic spectra.

1. Introduction

In this paper we continue our investigation of zero-temperature correlation functions of the XXZ Heisenberg model in the critical regime $-1 < \Delta < 1$ in an external magnetic field. The XXZ Hamiltonian is given by

$$\mathcal{H} = \sum_j \sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \Delta\ (\sigma_j^z \sigma_{j+1}^z - 1) - h\sigma_j^z \tag{1.1}$$

where the sum is over all integers j, L is the length of the lattice, σ^α are Pauli matrices and h is an external magnetic field. For later convenience we define $\Delta = \cos(2\eta)$, where $\frac{1}{2}\pi < \eta < \pi$. The free fermionic point in this notation is $\eta = \frac{3}{4}\pi$. The model (1.1) can be solved by means of the Bethe Ansatz, which yields a description of the spectrum and eigenstates (with N down spins and $L - N$ up spins) in terms of the roots of the following set of coupled algebraic equations [1, 2]:

$$\left(\frac{\sinh(\lambda_j - \mathrm{i}\eta)}{\sinh(\lambda_j + \mathrm{i}\eta)}\right)^L = -\prod_{k=1}^N \frac{\sinh(\lambda_k - \lambda_j + 2\mathrm{i}\eta)}{\sinh(\lambda_k - \lambda_j - 2\mathrm{i}\eta)} \qquad j = 1 \ldots N\,. \tag{1.2}$$

For the case $\Delta > -1$ it was proved by Yang and Yang in [2] that the ground state is characterized by a set of *real* λ_j subject to equations (1.2). In the thermodynamic limit the

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ground state is described by means of an integral equation for the density of the spectral parameters $\rho(\lambda)$:

$$2\pi\rho(\lambda) - \int_{-\Lambda}^{\Lambda} \mathrm{d}\mu\ K(\lambda,\mu)\ \rho(\mu) = \frac{-\sin(2\eta)}{\sinh(\lambda - \mathrm{i}\eta)\sinh(\lambda + \mathrm{i}\eta)}$$
$$\frac{\sin(4\eta)}{\sinh(\mu - \lambda + 2\mathrm{i}\eta)\sinh(\mu - \lambda - 2\mathrm{i}\eta)} = K(\lambda,\mu)\,. \tag{1.3}$$

Here the integration boundary Λ is a function of the magnetic field h. The physical picture of the ground state is that of a filled Fermi sea with boundaries $\pm\Lambda$. The dressed energy of a particle in the sea is given by the solution of the integral equation

$$\epsilon(\lambda) - \frac{1}{2\pi}\int_{-\Lambda}^{\Lambda} \mathrm{d}\mu\ K(\lambda,\mu)\ \epsilon(\mu)\ = 2h - \frac{2(\sin(2\eta))^2}{\sinh(\lambda - \mathrm{i}\eta)\sinh(\lambda + \mathrm{i}\eta)}\,. \tag{1.4}$$

The condition that the dressed energy vanishes at the Fermi edge $\epsilon(\pm\Lambda) = 0$ determines Λ as a function of the magnetic field h. It was shown in [2] that the ground state of (1.1) for $|\Delta| < 1$ is partially magnetized for magnetic fields $h < h_{\mathrm{c}} = 4\cos^2(\eta)$. For larger magnetic fields $h > h_{\mathrm{c}}$ the ground state is the saturated ferromagnetic state. For $h \to 0$ the integration boundary Λ tends to ∞, whereas for $h \to h_{\mathrm{c}}$ $\Lambda \to 0$. Below we only consider the region $h < h_{\mathrm{c}}$ and in particular the limiting case $h \to h_{\mathrm{c}}$.

The subject of this paper is the generating functional of correlation functions

$$G(m) = \langle \exp(\alpha Q_1(m))\rangle = \langle 0|\exp\left(\alpha\sum_{j=1}^{m}\frac{1-\sigma_j^z}{2}\right)|0\rangle \tag{1.5}$$

where $|0\rangle$ is the ground state.

Various correlation functions can be obtained from $\langle \exp(\alpha Q_1(m))\rangle$, (*e.g.*, the Ferromagnetic String Formation Probability [10] which corresponds to setting $\alpha = -\infty$) or

$$\langle 0|\sigma_m^z\sigma_1^z|0\rangle = 2\widehat{\Delta}\langle 0|\frac{\partial^2}{\partial\alpha^2}\bigg|_{\alpha=0}\exp(\alpha Q_1(m))|0\rangle + 1 - 4\int_{-\Lambda}^{\Lambda}\mathrm{d}\lambda\ \rho(\lambda) \tag{1.6}$$

where $\widehat{\Delta}$ is the lattice Laplacian acting on a function $f(j)$ defined on the lattice as $\widehat{\Delta}f(j) = f(j) + f(j-2) - 2f(j-1)$ and $\rho(\lambda)$ is defined in (1.3). In what follows we will consider $G(m)$ in the limit $h \to h_{\mathrm{c}}$. As will be shown below $G(m)$ is connected to the solution of a Painlevé V transcendent. The connection of Painlevé transcendents and integrable models with free-fermionic spectra is well established [3–5]. However, we want to emphasize that in the present case we are dealing with a theory of *interacting* fermions, so that the connection is novel.

2. Determinant representation

In a recent paper [6] we used the approach invented in [7, 8] (for a detailed exposition of this method see [9]) to derive the following representation of $G(m)$ in terms of the determinant of a Fredholm integral operator.

$$\langle 0|\exp(\alpha Q_1(m))|0\rangle = \frac{(\tilde{0}|\det\left(1+\widehat{V}\right)|0)}{\det\left(1-\widehat{K}/2\pi\right)}\,. \tag{2.1}$$

Here $\widehat{K}$ and $\widehat{V}$ are Fredholm integral operators with kernels given by (1.3) and

$$V(\lambda,\mu) = -\frac{\sin(2\eta)}{2\pi\sinh(\lambda-\mu)}\left\{\frac{1}{\sinh(\lambda-\mu+2\mathrm{i}\eta)} + \frac{e_2^{-1}(\lambda)e_2(\mu)}{\sinh(\lambda-\mu-2\mathrm{i}\eta)}\right.$$

$$+\mathrm{e}^{\alpha+\varphi_4(\mu)-\varphi_3(\lambda)}\left(\frac{1}{\sinh(\lambda-\mu-2\mathrm{i}\eta)}+\frac{e_1^{-1}(\mu)e_1(\lambda)}{\sinh(\lambda-\mu+2\mathrm{i}\eta)}\right)\Bigg\} \tag{2.2}$$

where

$$e_2(\lambda)=\left(\frac{\sinh(\lambda+\mathrm{i}\eta)}{\sinh(\lambda-\mathrm{i}\eta)}\right)^m \mathrm{e}^{\varphi_2(\lambda)} \qquad e_1(\lambda)=\left(\frac{\sinh(\lambda-\mathrm{i}\eta)}{\sinh(\lambda+\mathrm{i}\eta)}\right)^m \mathrm{e}^{\varphi_1(\lambda)}\,. \tag{2.3}$$

The quantities $\varphi_j(\lambda)$ are bosonic quantum fields [7] (they are called 'dual quantum fields') defined by

$$\varphi_a(\lambda)=p_a(\lambda)+q_a(\lambda) \qquad (\tilde{0}|q_a(\lambda)=0=p_a(\lambda)|0) \qquad (\tilde{0}|0)=1$$

$$[q_b(\mu),\,p_a(\lambda)]=\begin{pmatrix}1&0&1&0\\0&1&0&1\\0&1&1&1\\1&0&1&1\end{pmatrix}_{ab}\ln(h(\lambda,\mu))+\begin{pmatrix}1&0&0&1\\0&1&1&0\\1&0&1&1\\0&1&1&1\end{pmatrix}_{ab}\ln(h(\mu,\lambda)) \tag{2.4}$$

where $a,b=1,\ldots,4$, $h(\lambda,\mu)=\big(\sinh(\lambda-\mu+2\mathrm{i}\eta)/\mathrm{i}\sin(2\eta)\big)$. Note that the dual fields have the important property that they commute

$$[\varphi_a(\lambda),\varphi_b(\mu)]=0\,. \tag{2.5}$$

3. The limit of strong magnetic field

For large magnetic fields $h\to h_{\mathrm{c}}$, $h<h_{\mathrm{c}}$ the integration boundary Λ tends to zero according to

$$\Lambda=\tfrac{1}{2}|\tan(\eta)|\sqrt{h_{\mathrm{c}}-h}+\mathrm{O}(h_{\mathrm{c}}-h)\,. \tag{3.1}$$

This fact can be used to essentially simplify the above representation as was first done for a different correlation function in [10]. We first expand the kernel $V(\lambda,\mu)$ for small $|\lambda|,|\mu|\leqslant\Lambda$. We define $y=m|\cot(\eta)|$ as a shorthand notation. We then observe the following simplification of the commutation relations between the 'momenta' $p_a(\lambda)$ and 'coordinates' $q_a(\lambda)$:

$$[q_b(\mu),\,p_a(\lambda)]=\begin{pmatrix}0&0&1&-1\\0&0&-1&1\\-1&1&0&0\\1&-1&0&0\end{pmatrix}_{ab}\mathrm{i}\cot(2\eta)(\mu-\lambda)+\mathcal{O}(\Lambda^2)\,. \tag{3.2}$$

This allows us to reduce the number of dual fields from four to two via the identification $\varphi_2(\lambda)=-\varphi_1(\lambda)$ and $\varphi_4(\lambda)=-\varphi_3(\lambda)$. Furthermore the right-hand side of the commutators (3.2) is a linear function of $\lambda-\mu$ so that we can choose a representation such that $\varphi_a(\lambda)$ are linear functions in λ:

$$\varphi_a(\lambda)=\varphi_a+\varphi_a'\lambda \qquad p_a(\lambda)=p_a+p_a'\lambda \qquad q_a(\lambda)=q_a+q_a'\lambda \qquad a=1,3 \tag{3.3}$$

where $[p_a,q_b]=0$, $[p_a',q_b']=0$, and

$$[q_3,\,p_1']=-[q_1,\,p_3']=[q_1',\,p_3]=-[q_3',\,p_1]=-\mathrm{i}\cot(2\eta)\,. \tag{3.4}$$

As $\varphi_1(\lambda)$ appears in (2.2) only in the combination $\varphi_1(\lambda)-\varphi_1(\mu)$ the quantity φ_1 drops out. This in turn implies that p_3' and q_3' commute with all remaining operators and thus will not contribute to the expectation value with respect to $(\tilde{0}|$ and $|0)$. Therefore we can drop them everywhere. In the next step we perform a similarity transformation with $\exp(\mathrm{i}\lambda(y+\frac{1}{2}\mathrm{i}\varphi_1'))$, which leaves the determinant of the Fredholm integral operator invariant but brings the kernel to a more symmetric form, in which the dual fields now only enter via the expressions $\widehat{\alpha} := \alpha - 2\varphi_3$ and $\widehat{x} := y + \frac{1}{2}\mathrm{i}\varphi_1'$. In what follows it is crucial that the quantum operators in the dual bosonic Fock space $\widehat{x}$ and $\widehat{\alpha}$ are commuting objects and no problems with operator orderings occur before we evaluate the expectation value with respect to $(\tilde{0}|$ and $|0)$.

Now putting everything together we arrive at the following simplified representation valid in the limit $h \to h_c$:

$$\langle \exp(\alpha Q_1(m))\rangle = (\tilde{0}|\det\left(1+\widehat{V}_0\right)|0)\left(1+\frac{2\Lambda}{\pi}\cot(2\eta)+\mathcal{O}(\Lambda^2)\right) \tag{3.5}$$

where the kernel of $\widehat{V}_0$ is given by

$$V_0(\lambda,\mu) = (\exp(\widehat{\alpha})-1)\,\frac{\sin((\lambda-\mu)\widehat{x})}{\pi(\lambda-\mu)}\,. \tag{3.6}$$

Here we use the following notation for the dual fields:

$$\begin{aligned}
&\widehat{\alpha} = \alpha + \widehat{\alpha}_q + \widehat{\alpha}_p \qquad \widehat{x} = y + \widehat{x}_q + \widehat{x}_p \qquad (\tilde{0}|\widehat{x}_q = (\tilde{0}|\widehat{\alpha}_q = 0\\
&\widehat{x}_p|0) = \widehat{\alpha}_p|0) = 0 \qquad [\widehat{x}_q,\widehat{\alpha}_p] = [\widehat{\alpha}_q,\widehat{x}_p] = -\cot(2\eta) \qquad [\widehat{\alpha},\widehat{x}] = 0\,.
\end{aligned} \tag{3.7}$$

All other commutators vanish. We would like to emphasize that $\widehat{\alpha}$ and $\widehat{x}$ commute, which is important for the further analysis.

4. Connection with Painlevé V

Let us define a new variable $t = \Lambda\widehat{x}$ and consider the object

$$\sigma_0(t) = t\frac{\mathrm{d}}{\mathrm{d}t}\ln(\det(1+\widehat{V}_0))\,. \tag{4.1}$$

It was shown in [4] that σ_0 obeys a Painlevé V differential equation in the case where α and x are real numbers. In our case $\widehat{\alpha}$ and $\widehat{x}$ are quantum operators, but due to the fact that they are commuting we still can follow through the derivation of [4]. We thus find that $\sigma_0(t)$ obeys the following nonlinear differential equation:

$$\left(t\frac{\mathrm{d}^2\sigma_0}{\mathrm{d}t^2}\right)^2 = -4\left(t\frac{\mathrm{d}\sigma_0}{dt}-\sigma_0\right)\left(4t\frac{\mathrm{d}\sigma_0}{\mathrm{d}t}+\left(\frac{\mathrm{d}\sigma_0}{\mathrm{d}t}\right)^2-4\sigma_0\right) \tag{4.2}$$

which is identified (see [4]) as a τ-function form of the Painlevé V equation. Rewriting equations (4.2) in terms of the function $y_0(t)$ defined through

$$\sigma_0(t) = -4\mathrm{i}tu(t) + \frac{u^2(t)}{y_0(t)}(y_0(t)-1)^2 \qquad u(t) = \frac{4\mathrm{i}ty_0(t) - t\,\mathrm{d}y_0(t)/\mathrm{d}t}{2(y_0(t)-1)^2} \tag{4.3}$$

one obtains (again see [4]) the standard form of the Painlevé V differential equation for the function $\omega(t) = y_0(\frac{1}{2}t)$:

$$\frac{\mathrm{d}^2\omega}{\mathrm{d}t^2} = \left(\frac{\mathrm{d}\omega}{\mathrm{d}t}\right)^2\frac{3\omega-1}{2\omega(\omega-1)} + \frac{2\omega(\omega+1)}{\omega-1} + \frac{2\mathrm{i}\omega}{t} - \frac{1}{t}\frac{\mathrm{d}\omega}{\mathrm{d}t}\,. \tag{4.4}$$

The large t asymptotics of the solution of the above equations are known (see [5, 11–13]) and can be used to extract the large distance asymptotics of our Fredholm determinant.

Combining the results of [5] (asymptotics for σ_0) and [11, 13] (constant term) we obtain that

$$\ln(\det(1+\widehat{V}_0)) = \frac{2}{\pi}\Lambda\widehat{x}\widehat{\alpha} + \frac{\widehat{\alpha}^2}{2\pi^2}\ln(4\Lambda\widehat{x}) + 2\ln\left(g\left(\frac{\widehat{\alpha}}{2\pi}\right)\right) + \frac{\widehat{\alpha}^3}{8\pi^3\widehat{x}\Lambda}$$
$$-\frac{1}{\widehat{x}^2}\left[\frac{\widehat{\alpha}^2}{32\pi^2\Lambda^2}\cos(4\theta) + \frac{5\widehat{\alpha}^4}{128\pi^4\Lambda^2}\right] + \mathcal{O}\left(\frac{1}{\Lambda^3 y^3}\right) \tag{4.5}$$

where the distance y must be large (the product of y and the small parameter Λ should go to infinity) and where

$$g(\nu) = \mathrm{e}^{(1+\gamma)\nu^2}\prod_{n=1}^{\infty}\left(1+\frac{\nu^2}{n^2}\right)^n \mathrm{e}^{-\nu^2/n}$$
$$= \exp\left(\nu^2 - \frac{1}{2}\int_0^{\nu^2}\mathrm{d}t\ \left[\psi(1+\mathrm{i}\sqrt{t}) + \psi(1-\mathrm{i}\sqrt{t})\right]\right) \tag{4.6}$$
$$4\theta = 4\Lambda\widehat{x} + \frac{2}{\pi}\widehat{\alpha}\ln(4\Lambda\widehat{x}) - 4\arg\Gamma\left(\frac{\mathrm{i}\widehat{\alpha}}{2\pi}\right)$$

where $\psi(z) = \mathrm{d}\ln\Gamma(z)/\mathrm{d}z$ is the digamma function and γ is Euler's constant. Formula (4.6) for the phase θ of the corresponding solution of the Painlevé V equation (4.4) was also obtained in [12]. In order to obtain the large-distance asymptotics of the correlation function we still have to evaluate the expectation value in the dual bosonic Fock space. It is easiest to evaluate the quantities $\langle Q_1(m)^k\rangle = \langle \partial^k/\partial\alpha^k\big|_{\alpha=0}\mathrm{e}^{\alpha Q_1(m)}\rangle$ directly. Expanding

$$\exp\left(\frac{\widehat{\alpha}^2}{2\pi^2}\ln(4\Lambda\widehat{x})\right) = \exp\left(\frac{\widehat{\alpha}^2}{2\pi^2}\ln(4\Lambda y)\right)\left[1+\frac{\widehat{\alpha}^2}{2\pi^2}\frac{\widehat{x}_p+\widehat{x}_q}{y} + \mathcal{O}(y^{-2})\right]$$
$$\exp\left(\frac{\widehat{\alpha}^3}{8\pi^3\Lambda\widehat{x}}\right) = 1 + \frac{\widehat{\alpha}^3}{8\pi^3\Lambda y} + \mathcal{O}(\Lambda^{-1}y^{-2}) \tag{4.7}$$

we obtain the leading terms in the asymptotic decompositions as shown in the appendix:

$$\langle Q_1(m)\rangle\big|_{\text{lead}} = \frac{2\Lambda}{\pi}m|\cot\eta|\left(1+\frac{6\Lambda}{\pi}\cot(2\eta) + \mathcal{O}(\Lambda^2)\right)$$
$$\langle (Q_1(m))^2\rangle\big|_{\text{lead}} = \left[\left(\frac{2\Lambda}{\pi}m|\cot\eta|\right)^2 + \frac{\ln(4\Lambda m|\cot\eta|)+1+\gamma}{\pi^2}\right] \tag{4.8}$$
$$\times\left(1+\frac{8\Lambda\cot(2\eta)}{\pi} + \mathcal{O}(\Lambda^2)\right) + \frac{6\Lambda\cot(2\eta)}{\pi^3} + \mathcal{O}(\Lambda^2)\,.$$

Analogous formulae can be derived for $(Q_1(m))^k$ for $k > 2$. Using equation (1.6) we are now in a position to determine the large-distance asymptotics of the $\langle\sigma_m^z\sigma_1^z\rangle$ correlation functions. By acting with (twice) the lattice Laplacian on (4.8) we obtain

$$-\frac{2}{m^2\pi^2}\left(1+\frac{8\Lambda\cot(2\eta)}{\pi} + \mathcal{O}(\Lambda^2)\right). \tag{4.9}$$

This does not yet include the contribution from the cosine term in (4.5), which is very small as far as $\langle (Q_1(m))^2\rangle$ is concerned but becomes important upon differentiation. The leading contribution can be obtained by the methods explained in the appendix and is found to be

$$\left(\frac{2}{\pi^2} + \mathcal{O}(\Lambda)\right)\cos\left(4\Lambda m|\cot\eta|(1+\mathcal{O}(\Lambda))\right)\frac{1}{m^\theta} \tag{4.10}$$

where

$$\theta = 2 + \frac{8\Lambda\cot(2\eta)}{\pi} + \mathcal{O}(\Lambda^2). \tag{4.11}$$

Our final result for the first three terms of the asymptotics of the correlation function is thus

$$\langle\sigma_m^z\sigma_1^z\rangle = 1 - \frac{8\Lambda|\cot(\eta)|}{\pi} - \frac{2}{m^2\pi^2}\left(1 + \frac{8\Lambda\cot(2\eta)}{\pi}\right) + \frac{2}{\pi^2 m^\theta}\cos(4\Lambda m|\cot\eta|) + \ldots \tag{4.12}$$

where the errors in the Λ-expansion are given above. This agrees with the result obtained by means of finite-size corrections and conformal field theory.

5. Summary and conclusion

In this paper we have established a connection between the generating functional of correlation functions $G(m)$ (1.5) (and thus the correlator $\langle\sigma_m^z\sigma_1^z\rangle$) of the spin $\frac{1}{2}$ Heisenberg XXZ model in a magnetic field close to critical and the Painlevé V differential equation. Painlevé transcendents were known to describe correlation functions for models with free-fermionic spectra [3, 5]. For generic magnetic fields in the XXZ spin chain the general approach of [9] should be followed: the determinant representation (2.1), (2.2) should be used to embed the quantum correlation function in an integrable system of integro-difference equations and one then should solve the associated Riemann–Hilbert problem.

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Appendix

In this appendix we discuss how to evaluate the expectation value with respect to the dual quantum fields and explicitly evaluate the quantities $\langle(Q_1(m))^k\rangle$. According to (3.5)

$$\langle(Q_1(m))^k\rangle = \frac{(\tilde{0}|\partial^k/\partial\alpha^k\big|_{\alpha=0}\det\left(1+\widehat{V}_0\right)|0)}{1-(2\Lambda/\pi)\cot(2\eta)+\mathcal{O}(\Lambda^2)}\,. \tag{A.1}$$

Using equation (4.5) for the asymptotics of the logarithm of the determinant and expanding according to (4.7) we see that in order to get the leading asymptotics we need to evaluate the derivatives of expectation values of the form

$$(\tilde{0}|\exp\left(\frac{2}{\pi}\Lambda\widehat{x}\widehat{\alpha}\right)\exp\left(\frac{\widehat{\alpha}^2}{2\pi^2}\ln(4\Lambda y)\right)\left(g\left(\frac{\widehat{\alpha}}{2\pi}\right)\right)^2\left[1+\frac{\widehat{\alpha}^2}{2\pi^2}\frac{\widehat{x}_p+\widehat{x}_q}{y}\right]\left[1+\frac{\widehat{\alpha}^3}{8\pi^3\Lambda y}\right]|0)\,. \tag{A.2}$$

Using the commutation relations (3.7) we see that we can bring all terms into the form (we use that $(\tilde{0}|\widehat{x}_q = 0)$

$$(\tilde{0}|\widehat{x}_p^m e^{2\Lambda\widehat{x}\widehat{\alpha}/\pi}F(\widehat{\alpha})|0) \tag{A.3}$$

where $F(\widehat{\alpha})$ is only a function of $\widehat{\alpha}$ and contains no $\widehat{x}_p$'s or $\widehat{x}_q$'s. The central identities we will use in order to evaluate the expectation values are

$$z_m = (\tilde{0}|\widehat{x}_p^m \mathrm{e}^{2\Lambda\widehat{x}\widehat{\alpha}/\pi} F(\widehat{\alpha})|0)$$
$$= \frac{1}{\kappa}(\tilde{0}|\left(\frac{\widehat{x}_p}{\kappa} + \frac{2\Lambda\cot(2\eta)}{\pi\kappa}y\right)^m \exp\left(\frac{2\Lambda\alpha}{\pi\kappa}(y+\widehat{x}_p)\right)F(\widehat{\alpha})|0)\,. \tag{A.4}$$

where $\kappa = 1-(2\Lambda/\pi)\cot(2\eta)$. The identities are established via induction. The induction start $m=0$ is proved as follows. Expanding the exponential and using that $(\tilde{0}|\widehat{x}_q = 0$, $\widehat{\alpha}_p|0) = 0$ and $[\widehat{\alpha}_p, \widehat{\alpha}_q] = 0$ (to move all $\widehat{\alpha}_p$'s to the right) we obtain

$$z_0 = (\tilde{0}|\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{2\Lambda}{\pi}\right)^n (y+\widehat{x}_p)^n(\alpha+\widehat{\alpha}_q)^n F(\widehat{\alpha})|0)\,. \tag{A.5}$$

Expanding

$$(\tilde{0}|(y+\widehat{x}_p)^n(\alpha+\widehat{\alpha}_q)^n = \sum_{k=0}^{n}\frac{n!}{k!(n-k)!}(\tilde{0}|(y+\widehat{x}_p)^n\widehat{\alpha}_q^k\alpha^{n-k} \tag{A.6}$$

and then using the commutation relations

$$(\tilde{0}|[f(\widehat{x}_p), \widehat{\alpha}_q^k] = (\tilde{0}|(\cot(2\eta))^k f^{(k)}(\widehat{x}_p) \tag{A.7}$$

where $f^{(k)}$ is the kth derivative of the function f, we arrive at

$$z_0 = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\left(\frac{2\Lambda}{\pi}\right)^n\frac{n!(\cot(2\eta))^k\alpha^{n-k}}{k![(n-k)!]^2}(\tilde{0}|(y+\widehat{x}_p)^{n-k}F(\widehat{\alpha})|0)\,. \tag{A.8}$$

We now use the integral representation

$$\frac{1}{(n-k)!} = \frac{1}{2\pi\mathrm{i}}\oint \mathrm{d}t\,\frac{\mathrm{e}^t}{t^{n-k+1}} \tag{A.9}$$

where the integration contour is a small circle around the origin (and we integrate in the mathematically positive direction) in order to be able to perform the k-summation (which is of the form of a binomial sum)

$$z_0 = \frac{1}{2\pi\mathrm{i}}\oint\mathrm{d}t\,\frac{\mathrm{e}^t}{t}\sum_{n=0}^{\infty}\left(\frac{2\Lambda}{\pi}\right)^n(\tilde{0}|[\cot(2\eta)+(\alpha/t)(y+\widehat{x}_p)]^n F(\widehat{\alpha})|0)\,. \tag{A.10}$$

The n-summation can be performed using $(1-z)^{-1} = \sum_{k=0}^{\infty} z^k$. Finally we perform the t-integration formally using the identity

$$\frac{1}{2\pi\mathrm{i}}\oint\mathrm{d}t\,\frac{\mathrm{e}^t}{t-O(\alpha)} = \mathrm{e}^{O(\alpha)} \tag{A.11}$$

where $O(\alpha)$ is an operator depending on α. Here we need to keep in mind that we are interested in evaluating derivatives with respect to α at $\alpha = 0$. This yields the result (A.4) for $m = 0$. The induction step goes as follows. We rewrite z_m as

$$z_m = (\tilde{0}|\widehat{x}_p^{m-1}[\widehat{x}_p, \mathrm{e}^{2\Lambda\widehat{x}\widehat{\alpha}/\pi}F(\widehat{\alpha})]|0) \tag{A.12}$$

where we used that $\widehat{x}_p|0) = 0$. Evaluating the commutator using

$$\widehat{x}_p f(\widehat{\alpha})|0) = \cot(2\eta)f'(\widehat{\alpha})|0) \tag{A.13}$$

where f' is the derivative of f and collecting terms we obtain

$$z_m = \frac{1}{\kappa}(\tilde{0}|\widehat{x}_p^{m-1}\mathrm{e}^{2\Lambda\widehat{x}\widehat{\alpha}/\pi}\left[\cot(2\eta)F'(\widehat{\alpha}) + \frac{2\Lambda\cot(2\eta)}{\pi}yF(\widehat{\alpha})\right]|0)\,. \tag{A.14}$$

Using the induction assumption and again (A.13) then yields the desired result (A.4). Let us now demonstrate how to evaluate the leading contributions to (A.2) for $\langle Q_1(m)\rangle$ and $\langle (Q_1(m))^2\rangle$. They are given by

$$\langle (Q_1(m))^l\rangle\big|_{\text{lead}} = \frac{\partial^l}{\partial\alpha^l}\bigg|_{\alpha=0} (\tilde{0}|\mathrm{e}^{2\Lambda\widehat{x}\widehat{\alpha}/\pi} \exp\left(\frac{\widehat{\alpha}^2}{2\pi^2}\ln(4\Lambda y)\right)\left(g\left(\frac{\widehat{\alpha}}{2\pi}\right)\right)^2 |0)$$
$$\times\left(1+\frac{2\Lambda}{\pi}\cot(2\eta)+\mathcal{O}(\Lambda^2)\right). \tag{A.15}$$

We apply (A.4) with $m = 0$ and $F(\alpha) = \mathrm{e}^{\widehat{\alpha}^2\ln(4\Lambda y)/2\pi^2}(g(\widehat{\alpha}/2\pi))^2$, then perform the differentiations with respect to α and set α to zero, and finally use (A.13) to evaluate the expectation value using the fact that $(\tilde{0}|\widehat{\alpha}_q = 0$, $\widehat{\alpha}_p|0) = 0$. The function g has the properties that $g(0) = 1$, $g'(0) = 0$ and $g''(0) = 2(1+\gamma)$, where γ is Euler's constant, which leads to the result

$$\langle Q_1(m)\rangle\big|_{\text{lead}} = \frac{2\Lambda}{\pi}m|\cot\eta|\left(1+\frac{6\Lambda}{\pi}\cot(2\eta)+\mathcal{O}(\Lambda^2)\right)$$
$$\langle (Q_1(m))^2\rangle\big|_{\text{lead}} = \left(\left(\frac{2\Lambda}{\pi}m|\cot\eta|\right)^2+\frac{\ln(4\Lambda m|\cot\eta|)+1+\gamma}{\pi^2}\right)$$
$$\times\left(1+\frac{8\Lambda\cot(2\eta)}{\pi}+\mathcal{O}(\Lambda^2)\right). \tag{A.16}$$

The contributions of the subleading terms can be taken into account in an analogous way, which leads to the results (4.8). We note that contributions from further subleading terms are of higher order in Λ and y.

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